Various proofs of 0.999... $= 1$

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Contents

1 Algebraic Proof

1. Let
$$x = 0.\overline{9} \tag{1}$$
where $0.\overline{9}$ denotes $0.999\ldots$.

2. Multiply both sides by 10:
$$10x = 9.\overline{9} \tag{2}$$

3. Subtract (1) from (2):
$$\begin{aligned} 10x - x &= 9.\overline{9} - 0.\overline{9} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

4. Since $x = 0.\overline{9}$ (from (1)), we conclude:
$$0.\overline{9} = 1 \quad \square$$

2 Infinite Series Proof

We observe that
$$0.\overline{9} = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots := S,$$
where $0.\overline{9}$ denotes $0.999\ldots$. This is a geometric series S with first term $a = 0.9$ and common ratio $r = 0.1$.
The sum of an infinite geometric series is given by
$$S = \frac{a}{1-r}. \tag{3}$$
Substituting $a = 0.9$ and $r = 0.1$ into (3), we find
$$S = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1 \quad \square$$

Proof of the Geometric Series Formula

For completeness, we prove (3). Consider a geometric series of the form:
$$S = a + ar + ar^2 + ar^3 + \cdots, \tag{4}$$
where a is the first term and r is the common ratio ($|r| < 1$ for convergence).
Multiply both sides of (4) by r:
$$Sr = ar + ar^2 + ar^3 + ar^4 + \cdots. \tag{5}$$
Subtract (5) from (4). Notice that most terms on the right-hand side cancel:
$$S - Sr = a.$$
Factor out S on the left-hand side:
$$S(1-r) = a.$$
Solving for S, we obtain
$$S = \frac{a}{1-r} \quad \square$$

3 Archimedean Property

1. Assume for contradiction that $0.999\ldots \neq 1$. Then $0.999\ldots < 1$.

2. Let $\varepsilon = 1 - 0.999\ldots$, which is a positive real number.

3. By the Archimedean Property, there exists a positive real integer N such that $10^{-N} < \varepsilon$.

4. Consider the partial sum of $0.999\ldots$ after N decimal places, which is $1 - 10^{-N}$. Since all subsequent digits are $9s$, $0.999\ldots \geq 1 - 10^{-N}$.

5. $\Rightarrow \quad 1 - 0.999\ldots \leq 10^{-N}$.

6. Combining the results from steps 3 and 5, we get $\varepsilon = 1 - 0.999\ldots \leq 10^{-N} < \varepsilon$.
   $\Rightarrow \quad \varepsilon < \varepsilon$
   $\Rightarrow \quad 0.999\ldots = 1 \quad \square$